UNIVERSIDAD CARLOS III DE MADRID working papers

UC3M Working papers
Economics
17-17
December, 2017
ISSN 2340-5031

Departamento de Economía
Universidad Carlos III de Madrid
Calle Madrid, 126
28903 Getafe (Spain)
Fax (34) 916249875

Two-sided strategy-proofness in many-to-many matching markets

Antonio Romero-Medina[*], Matteo Triossi [‡]

December 18, 2017

Abstract

We study the existence of group strategy-proof stable rules in many to-many matching markets. We show that when firms have acyclical preferences over workers the set of stable matchings is a singleton, and the worker-optimal stable mechanism is a stable and group strategy-proof rule for firms and workers. Furthermore, acyclicity is the minimal condition guaranteeing the existence of stable and strategy-proof mechanisms in many-to-many matching markets.

JEL Classification Numbers: C71; C78; D71; D78.
Keywords: Many-to-many markets, Acyclicity, Stability, Two-sided Group Strategy-proofness, Singleton core.

[*] Departamento de Economía, Universidad Carlos III de Madrid.
[†] Centro de Economía Aplicada, Departamento de Ingeniería Industrial, Universidad de Chile. Beaucheff 851, Santiago, Chile.
[‡] Corresponding Author

1 Introduction

Many relevant real-world markets are many-to-many. The canonical example of a many-to-many market is the specialty training followed by junior doctors in the UK (Roth, 1991). Other examples of many-to-many markets are markets where workers are allowed to work part-time and the non-exclusive dealings between down-stream firms and up-stream providers. Many-to-many markets are also useful to model multi-unit assignment problems such as course allocations (see Budish 2011, Sönmez and Ünver, 2010, Kojima, 2013) or the assignment of landing slots (see Schummer and Abizada, 2017, Schummer and Vohra, 2013).

In many-to-many matching markets, no stable and strategy-proof mechanism exists, even for agents on one side of the market (see Roth and Sotomayor, 1990). Furthermore, even in one-to-one markets, there is no mechanism that is stable and strategy-proof for the agents on both sides of the market. Due to these negative results, the literature has concentrated on mechanisms guaranteeing strategy-proofness on one side of the market, thus overlooking preference manipulation from agents on the other side of the market (but see Romero-Medina and Triossi, 2013a for capacity manipulation). However, manipulation by agents on both sides of the market is a concern, for example, in different student assignment problems (see Abdulkadiroğlu et al., 2005 and Figueroa et al, 2017).

In this paper, we explore the possibility of designing revelation mechanisms that are stable, strategy-proof and group strategy-proof for agents on both sides of the market. Indeed, stability and strategy-proofness are central concerns in market design. Theoretical and empirical findings suggest that markets that achieve stable outcomes are more successful than markets that do not achieve stable outcomes (see Roth and Sotomayor, 1990; Abdulkadiroğlu and Sönmez, 2013).[1] Additionally, strategy-proofness prevents agents from needing to strategize. This is relevant in markets where agents have little information or differ in their sophistication. Finally, group strategy-proofness implies strategy-proofness and prevents welfare losses resulting from collusion among agents.

We show that if the firms have acyclical preferences, the worker-optimal stable mechanism is group strategy-proof. A cycle in the preferences of the firms occurs when there is an alternating list of firms and workers "on a circle" such that every firm prefers the worker on its clockwise side to the worker on its counterclockwise side and finds both acceptable. We say that preferences are acyclical if there are no cycles.

First, we show that if the preferences of the firms are acyclical, the set of stable matchings is a singleton, and the unique stable matching can be implemented through a procedure that we call Adjusted Serial Dictatorship. In this procedure, each worker, at her turn, selects her favorite firms among those she

[1] Moreover, in the school assignment model and the course allocation problem, stability embodies a notion of fairness because it eliminates justified envy, that is, situations in which an agent prefers to receive another assignment over one of her assignments and has a higher priority at the preferred assignment (see Balinski and Sönmez, 1999; Sönmez, and Abdulkadiroğlu, 2003).

is acceptable to and that still have vacant positions. We employ this result to show that under acyclicity, any stable mechanism is group strategy-proof both for firms and workers. We conclude by showing that acyclicity is also the minimal condition guaranteeing the existence of a mechanism that is stable and strategy-proof for agents on both sides of a many-to-many matching market. More precisely, we show that if the preferences of the firms have a cycle, there exists a profile of preferences for the workers such that no stable mechanism is strategy-proof. Our results imply that strategy-proofness and group strategy-proofness are equivalent requirements when imposed on a stable mechanism. In general, group strategy-proofness is more demanding than strategy-proofness. In particular, in the school assignment model, the student-optimal stable mechanism always provides a stable and strategy-proof assignment. However, efficiency and group strategy-proofness require priorities to satisfy an acyclicity condition (see Ergin, 2002). Our characterization contributes in explaining the restrictiveness of imposing strategy-proofness on stable mechanisms in many-to-many matching markets.

Finally, we apply our results to the course allocation problem. In this case, only one side of the market is strategic, and acyclicity is a sufficient condition for the existence of a strategy-proof mechanism. We show that acyclicity is also necessary if the designer cannot condition the mechanism on the capacities of the courses.

1.1 Related literature

Acyclical preferences have been extensively studied in matching markets. The concept of acyclicity that we use coincides with that introduced in Romero-Medina and Triossi (2013b) for one-to-one matching markets. Ergin (2002) introduces a weaker notion of acyclicity and shows that the worker-optimal stable mechanism is efficient and group strategy-proof if and only if the preferences of the firms are acyclical. Kesten (2012) and Romero-Medina and Triossi (2013a) find that two different forms of acyclicity are necessary and sufficient conditions for worker-optimal stable matching to be immune from capacity manipulation.

Our results complement those in Jiao and Tian (2017) and Kojima (2013). The first paper proves that the worker-optimal stable mechanism is group strategy-proof for workers if preferences satisfy the extended max-min criterion and a quota saturability condition. The preference domain in Jiao and Tian (2017) reflects a high degree of ambiguity aversion in agents. Instead, we assume the preferences of the firms to be responsive.[2] In a multi-unit assignment problem, Kojima (2013) proves that the worker-optimal stable matching is strategy-proof for workers if and only if any cycle involves only the top-ranked workers, a condition that he calls essential homogeneity, which is weaker than the concept of acyclicity that we employ in this paper. Romero-Medina and Triossi (2017) show that the equivalence between strategy-proof and group strategy-proof also holds in the multi-unit assignment problem. In contrast to the previously men-

[2] The two domains are unrelated.

tioned papers, we focus on preventing manipulation and collusion by the agents on both sides of the markets.

The structure of this paper is as follows. Section 2 introduces the model. Section 3 presents the results. Section 4 concludes.

2 The model

In our model, there are two disjoint and finite sets of agents, the set of workers W and the set of firms F. A generic worker will be denoted by w, a generic firm by f and a generic agent by $v \in V = F \cup W$. Each worker can work for more than one firm, and firms can hire more than one worker. Let $P_F = (P_f)_{f \in F}$ be a list of firms' preferences over subsets of workers, where for every $f \in F$, P_f is a strict order defined on 2^W. For all $w, w' \in W$, $wP_f w'$, $wP_f \varnothing$ and $\varnothing P_f w$ denote $\{w\} P_f \{w'\}$, $\{w\} P_f \varnothing$ and $\varnothing P_f \{w\}$, respectively. Let $P_W = (P_w)_{w \in W}$ be a list of workers' preferences over subsets of firms, where for every $w \in W$, P_w is a strict order defined on 2^F. For each $v \in V$, we denote by R_v the corresponding weak preferences. A profile $P = (P_v)_{v \in V}$ is a list of preference orderings. Given a profile $P = (P_v)_{v \in V}$ and $V' \subseteq V$, we denote by $P_{V'}$ the vector $(P_v)_{v \in V'}$. The triple (F, W, P) is called a **matching market**. The favorite group of workers for firm f among those belonging to W' is called the **choice set from** W'. We denote the choice set from W' by $Ch_f(W', P_f)$ or by $Ch_f(W')$ when no ambiguity is possible. Formally, $Ch_f(W', P_f) = \max_{P_f} \{W'' : W'' \subset W'\}$. If $\varnothing P_f W'$ firm f prefers not to employ any worker rather than jointly employing the workers in W', then W' is called **unacceptable** to f. Otherwise W' is **acceptable** to f. We denote the set of workers who are individually acceptable to f by $A(f, P_f)$ or $A(f)$ when no ambiguity is possible. The maximum number of workers that firm f is willing to hire is f's **capacity,** which we denote by q_f; formally, $q_f = \max \{|W'| : W' P_f \varnothing\}$.[3] For every $w \in W$ and for every $F' \subseteq F$, we define $Ch_w(F', P_w)$, $Ch_w(F')$, $A(w)$, and q_w similarly.

Matchings assign workers to firms. A **matching** on (F, W, P) is a function $\mu : V \to 2^V$ such that, for every $(f, w) \in F \times W$: (i) $\mu(f) \in 2^W$, (ii) $\mu(w) \in 2^F$ and (iii) $f \in \mu(w) \Leftrightarrow w \in \mu(f)$. We denote by $\mathcal{M}$ the set of matchings on (F, W, P). A matching μ is **individually rational** in (F, W, P) , if $Ch_v(\mu(v)) = \mu(v)$ for all $v \in V$. Individual rationality captures the idea that hiring and joining a firm are voluntary. A matching μ is **blocked by the pair** $(f, w) \in F \times W$, $f \notin \mu(w)$, if (i) $f \in Ch_h \mu(w) \cup \{f\}$ and (ii) $w \in Ch_f (\mu(f) \cup \{w\})$. A firm-worker pair (f, w) blocks a matching μ if worker w is not employed at f, but she would like to join f eventually after leaving some of her current jobs, and f would like to hire w eventually after firing some of its current employees. A matching μ is **stable** in (F, W, P) if it is individually rational and if no pair blocks it. Otherwise, μ is **unstable**. $\Gamma(F, W, P)$ denotes the set of matchings that are stable in (F, W, P).

The set of stable matchings may be empty. For this reason, we focus on responsive preferences that guarantee that the set of stable matchings is nonempty.

[3]For every set S, the symbol $|S|$ denotes the cardinality of S.

We say that the preferences of a firm, P_f, are **responsive** if, for all $W' \subset W$ such that $|W'| \leq q_f - 1$ and for all $w, w' \in W$: (i) $W' \cup \{w\} \, P_f W' \cup \{w'\} \Leftrightarrow w P_f w'$ and (ii) $W' \cup \{w\} \, P_f W' \Longleftrightarrow w \in A(f)$. In words, f has responsive preferences if for any two assignments that differ in one worker only, it prefers the assignment containing the most preferred worker. We denote by $\mathcal{P}$ the set of responsive preferences. Responsive preferences for workers are defined similarly. The set of responsive preferences is denoted by $\mathcal{P}$. If firms and workers have responsive preferences, the set of stable matchings forms a nonempty lattice (see Alkan, 1999). Furthermore, there exists a stable matching that is Pareto superior for workers to all other stable matchings, the **worker-optimal stable matching,** which we denote by $\mu^W(P)$.

A **cycle** (of length $T+1$) in P_F is given by distinct workers $w_0, w_1, ..., w_T \in W$ and distinct firms $f_0, f_1, ..., f_T \in F$ such that

1. $w_T P_{f_T} w_{T-1} P_{f_{T-1}}, ..., P_{f_2} w_1 P_{f_1} w_0 P_{f_0} w_T$;

2. for every t, $0 \leq t \leq T$, $w_t \in A(f_{t+1}) \cap A(f_t,)$, where $w_{T+1} = w_0$.

A responsive preference profile on individual workers P_F is **acyclical** if it has no cycles.

Let us assume that a cycle exists. If every worker w_{t-1} is initially assigned to firm f_t, every firm is willing to exchange its assigned worker with its successor w_t.

A mechanism φ is a function that associates a matching to every preference profile within a given domain $\mathcal{D} \subseteq \mathcal{P}^{|V|}$: $\varphi : \mathcal{D} \to \mathcal{M}$. A mechanism φ is *stable* if $\varphi(P)$ is stable for all $P \in \mathcal{D}$. The worker-optimal stable mechanism defined by $\varphi(P) = \mu^W(P)$ is an example of stable mechanism. A mechanism φ is **Pareto optimal** if for every $P \in \mathcal{D}$, there exists no individually rational matching μ, such that $\mu(v) \, R_v \varphi(P)(v)$ for every $v \in V$ and $\mu(v) \, P_v \varphi(P, q)(v)$ for at least one v. A mechanism is Pareto optimal if it implements matchings for which there is no alternative individually rational matching that is weakly preferred by all agents and strongly preferred by at least one agent. A mechanism φ is **strategy-proof** if for every $v \in V$, $\varphi(P)(v) \, R_v \varphi(P'_v, P_{-v})(v)$ for every $P \in \mathcal{D}$, $P'_v \in \mathcal{P}$. A mechanism is strategy-proof if reporting her true preference relation is a (weakly) dominant strategy for every agent. A mechanism φ is **group strategy-proof** if there does not exist $P \in \mathcal{D}$, a nonempty set of agents, $V' \subset V$, P, $P'_{V'} = (P'_v)_{v \in V'} \in \mathcal{P}^{|V'|}$ such that $\varphi(P'_{V'}, P_{V \setminus V'})(v) \, R_v \varphi(P)(v)$ for every $v \in V'$ and $\varphi(P'_{V'}, P_{V \setminus V'})(v') \, P_{v'} \varphi(P)(v')$ for some $v' \in V'$. The mechanism φ is group strategy-proof if no subset of agents can benefit by jointly misrepresenting their preferences. Since capacity, in our model, is endogenous to the preference profile, a strategy-proof mechanism prevents capacity manipulation (see Sönmez, 1997).

The concept of group strategy-proofness that we employ is stronger than group incentive compatibility, which is also referred to as weak group strategy-proofness. The latter requires that no coalition of agents can misrepresent their preferences in a way that makes each member of the coalition strictly better off (see Roth and Sotomayor, 1990; Hatfield and Kojima, 2009; Barberá et al., 2016). The concept of group strategy-proofness in Jiao and Tian (2017) coincides with group incentive compatibility and restricts its attention to group deviations by the agents on one side of the market only.

3 Group strategy-proofness and uniqueness

In many-to-many matching markets, if the preferences of both workers and firms are responsive, no stable mechanism is strategy-proof or Pareto optimal for workers. In this section, we prove that the assumption of acyclical preferences is a necessary and sufficient condition to overcome the incompatibility of strategy-proofness, stability and Pareto optimality.

We first show that when the firms have acyclical preferences over individual workers, there exists an underlying order w_1,w_2,...,$w_{|W|}$ on the set of workers that is able to sustain a stable matching through an adjusted serial dictatorship.

Let us assume that P_F is acyclical and define the following order on W. Let $w_1 \in W$ be a worker who is never ranked below first place by any firm to which she is acceptable. Formally, let w_1 be such that there exist no $f \in F$, $w \in W$ with $wP_f w_1$ and $w_1 P_f \emptyset$. Such a w_1 exists because P_F is acyclical. For $0 \leq t \leq |W| - 1$, let w_{t+1} be a worker who is never ranked below workers other than $w_1, w_2, ..., w_t$ by any firm to which she is acceptable. Formally, let $w_{t+1} \in W$ be such that there exist no $f \in F$, $w \in W \setminus \{w_1, w_2, ..., w_t\}$ such that $wP_f w_{t+1}$ and $w_{t+1} P_f \emptyset$. Such a w_{t+1} exists because P_F is acyclical.

Next, we define an **Adjusted Serial Dictatorship** by letting each worker choose among the firms that she is acceptable to and that still have vacant positions according to w_1,w_2,...,$w_{|W|}$.

Let $A_1(P) = \{f : w_1 \in A(f)\}$, be the set of firms to which worker w_1 is acceptable. Define $\mu(P)(w_1) = C_{w_1}(A_1(P))$. For all t, $1 \leq t \leq |W| - 1$, let $A_{t+1}(P) = \left\{f : w_{t+1} \in A(f), \bigcup_{s \leq t, f \in \mu(P)(w_s)} \{w_s\} < q_f\right\}$, be the set of firms worker w_{t+1} is acceptable to and that have vacant positions. Define $\mu(P)(w_{t+1}) = C_{w_{t+1}}(A_{t+1}(P))$. For every $f \in F$, let $\mu(f) = \bigcup_{f \in \mu(w)} \{w\}$.[4] First, we prove that matching $\mu(P)$, the outcome of the Adjusted Serial Dictatorship, is the unique stable matching of market (F, W, P, q).

Proposition 1 *Let $M = (F, W, P, q)$ be a matching market and let P_F be acyclical. Matching $\mu(P)$ is the unique stable matching of market (F, W, P).*

The result holds without any assumption on the preferences of the workers and implies the existence of a stable matching whenever the preferences of the firms are responsive and acyclical, regardless of the preferences of the workers.

Next, we assume that the preferences of the workers are responsive and prove our main result: no coalition of agents can benefit from preference manipulation

[4]Notice that the selection of w_t is not unique, for every t, $1 \leq t \leq |W| - 1$ and thus, the procedure defines a family of orders on W. In particular, the notation $\mu(P)$ could be be ambiguous. Proposition 1 implies that all such orders generate the same stable matching; thus, they are equivalent.

if the worker-optimal stable mechanism $\mu^W(P)$ is used.[5]

Theorem 1 *Let $M = (F, W, P)$ be a matching market, let P_W be responsive, and let P_F be acyclical. Then, the worker-optimal stable mechanism, $\mu^W(P)$ is group strategy-proof.*

The proof of Theorem 1 is based on the characterization of the worker-optimal stable matching provided in Proposition 1 and the observation that the outcome of any deviation can be reached through a deviation that preserves the acyclicity of the preferences of the firms.

Ergin's acyclicity (see Ergin, 2002) prevents the coalitional deviation of workers in many-to-one matching markets. Essential homogeneity (see Kojima, 2013) prevents individual manipulation of the workers. Acyclicity simultaneously prevents individual and coalitional deviations of both firms and workers.

From Theorem 1, easily follows the Pareto optimality of the worker-optimal stable mechanism.

Corollary 1 *Let $M = (F, W, P)$ be a matching market, let P_W be responsive, and let P_F be acyclical. Then, the worker-optimal stable mechanism $\mu^W(P)$ is Pareto optimal, Pareto optimal for workers and Pareto optimal for firms.*

Next, we study whether it is possible to weaken the acyclicity requirement and find a mechanism that is stable and strategy-proof for all agents. First we show that without acyclicity, the worker-optimal stable mechanism is not strategy-proof for firms.

Lemma 1 *Assume that P_F has a cycle. Then, there exists a profile of responsive preferences for workers P_W such that the worker-optimal stable mechanism is not strategy-proof for firms.*

The intuition behind Lemma 1 is that if the preferences of the firms are not acyclic, there exists a profile of preferences for workers such that the resulting market has two stable matchings. In this case, any firm f preferring the firm-optimal stable matching to the worker-optimal stable matching can successfully manipulate the worker-optimal stable mechanism.

Thus, a singleton core is necessary for the existence of a stable and strategy-proof mechanism. However, having a unique stable matching is not sufficient for the existence of a stable and strategy-proof mechanism. In the next example, we provide a market with a singleton core where an agent can successfully manipulate the unique stable matching because there is a cycle in the preferences of the firms.

Example 1 *Let us assume $F = \{f_1, f_2, f_3\}$ and $W = \{w_1, w_2, w_3\}$. Set P_{w_1} : $\{f_1, f_2\}, \{f_2\}, \{f_1\}$, $P_{w_2} : \{f_3\}, \{f_2\}$ and $P_{w_3} : \{f_1\}, \{f_3\}$. Let $w_1 P_{f_1} w_3 P_{f_3} w_2 P_{f_2} w_1$,*

[5] Without making any assumption on workers preferences, if the preferences of the firms are acyclical, the worker-optimal stable mechanism is group strategy-proof for workers. The easy proof is available upon request.

$A(f_1) = \{w_1, w_3\}$, $A(f_2) = \{w_1, w_2\}$, $A(f_3) = \{w_2, w_3\}$, and let $q_f = 1$ for all $f \in F$.
There exists a unique stable matching μ where $\mu(f_i) = \{w_i\}$ for $i = 1, 2, 3$. If any stable mechanism is employed and worker w_1 reports preferences $P'_{w_1} = \{f_2\}$, she obtains a position at f_2, which she strictly prefers to f_1.

The intuition provided by Example 1 and Lemma 1 leads us to prove that acyclicity is the minimal condition guaranteeing the existence of a stable group strategy-proof mechanism.

Proposition 2 *Assume that P_F has a cycle. Then, there exist a profile of responsive preferences for workers P_W such that no stable mechanism is strategy-proof.*

In conclusion, we can integrate the main findings of the Section in the following theorem.

Theorem 2 *The following statements are equivalent:*

1. *There exists a stable and strategy-proof mechanism.*
2. *There exists a stable and group strategy-proof mechanism.*
3. *P_F is acyclical.*

3.1 Course allocation

Next, we apply our results to the case where firms are objects to be consumed. Thus, their preferences are to be intended as priorities. This problem is usually called the course allocation problem. It is a one-sided multi-unit assignment problem under priorities, in which only workers are strategic. Formally, a course allocation problem can be identified with a matching market (F, W, P_F, P_W), where F, or the set of firms, is identified with the set of courses to be distributed among the workers W, who now play the role of students. The preferences of the firms are to be interpreted as priorities. For every $f \in F$, let $\succ_f$ be the restriction of P_f to individual students. Formally, for every $w, w' \in W \cup \{\emptyset\}$, let $w \succ_f w'$ if and only if $w P_f w'$. The set of stable matchings depends only on the preferences of the workers, on the preferences of the firms over individual workers, and on the capacities of the firms. We denote a course allocation problem as by $(F, W, \succ_F, P_W, q)$, where $\succ_F = (\succ_f)_{f \in F}$ and $q_F = (q_f)_{f \in F}$. We call $(\succ_F, q)$ a **priority structure**.

The priority structure $(\succ_F, q)$ satisfies **essential homogeneity**[6] if there are no $f_0, f_1, ..., f_T \in F$, $w_0, w_1, ..., w_T \in W$ and $W_0, ..., W_T \subseteq W \setminus \{w_0, w_1, ..., w_T\}$:

1. $w_T \succ_{f_T} w_{T-1} \succ_{f_{T-1}} ... \succ_{f_2} w_1 \succ_{f_1} w_0 \succ_{f_0} w_T$;

[6]This definition adapts that proposed in Kojima (2013), accounting for situations where courses have different eligibility requirements.

2. for every t, $0 \leq t \leq T$, $w_t \in A(f_{t+1}) \cap A(f_t,)$,where $f_{T+1} = f_0$.

3. for every t, $0 \leq t \leq T$, $|W_{t+1}| = q_{t+1} - 1$ and $w \succ_{f_{t+1}} w_t$ for each $w \in W_i$ where $f_{T+1} = f_0$.

From the proof of Theorem 1 in Kojima (2013), it follows that essential homogeneity is equivalent to the existence of a stable mechanism that is strategy-proof for students.[7] Essential homogeneity is weaker than acyclicity but it does not guarantee that the set of stable matchings is a singleton (see Romero-Medina and Triossi, 2013b), nor the existence of a mechanism that is stable and strategy-proof for the agents on the two sides of the market.

Acyclicity is a more than necessary condition for the existence of a stable mechanism that is strategy-proof for workers when capacities are given. However, in several practical course allocation problems, capacities are decided year by year depending on infrastructure and expected demand. In such situations, only $\succ_F$ can be assumed as given, and the objective of the designer is to devise a strategy-proof mechanism that works for any capacity vector $q = (q_f)_{f \in F}$. We next prove that this is possible only if $\succ_F$ is acyclical.

Lemma 2 *Assume that $\succ_F$ has a cycle. Then, there exists a profile of responsive preferences for firms P_F and a vector of capacities q such that no stable mechanism is strategy-proof for workers.*

Thus acyclicity is a minimal condition on priorities that guarantees strategy-proofness for any vector of capacity q. The result suggests that if a revelation mechanism is to be used, the use of acyclical priorities is the only choice that guarantees stability of the assignment and truthful behavior by students.

Proposition 3 *The following statements are equivalent:*

1. *There exists a stable mechanism that is strategy-proof for workers for every $q = (q_f)_{f \in F}$.*

2. *There exists a stable mechanism that is group strategy-proof for workers for every $q = (q_f)_{f \in F}$.*

3. *$\succ_F$ is acyclical.*

The proof easily follows directly from Theorem 1 and Lemma 2.

4 Conclusions

In this paper, we prove that the worker-optimal stable mechanism is group strategy-proof and Pareto optimal in many-to-many matching markets if the preferences of the firms are acyclical. In this case, the unique stable matching can be obtained through an Adjusted Serial Dictatorship.

[7] Romero-Medina and Triossi (2017) prove that essential homogeneity is also equivalent to the existence of a stable mechanism that is group strategy-proof for students.

The restriction to acyclical priorities is particularly appropriate in situations where preferences reflect an underlying merit-based ranking. In these cases, a serial dictatorship is an appealing implementation mechanism (see also Ehlers and Klaus, 2003).

Acyclicity is the minimal condition guaranteeing the existence of a strategy-proof stable mechanism in many-to-many markets. This result can be interpreted as negative because acyclicity is a strong restriction. This interpretation suggests that whenever the restriction of acyclical preferences or priorities is not deemed reasonable or appropriate, the designer should explore alternative options. A first alternative is weakening the equilibrium requirements by using a non-revelation mechanism. A second possibility is to weaken the stability requirement on the mechanism.

Appendix

Proof of Proposition 1. First, we show that any matching μ obtained from an adjusted serial dictatorship is stable. The definition of μ implies that it is individually rational. Next, we prove by contradiction that there is no pair blocking μ. Assume that there exists a pair $(f, w) \in F \times W$ blocking μ. Let $s \in \{1, ..., |W|\}$ such that $w = w_s$. We have $\mu(w_s) = C_{w_s}(A_s(P))$. First, assume that $|\mu(f)| < q_f$. Then, $f \in A_s$, yielding a contradiction. Second, consider the case in which $|\mu(f)| = q_f$. Because (f, w_s) blocks μ, $w_s P_f w'$ for some $w' \in \mu(f)$. From the definition of the sequence $w_1, w_2, ..., w_{|W|}$, it follows that $w' = w_l$ for some $l > s$. Thus, $f \in A_s(P)$, yielding a contradiction.
Finally, we prove that there is exactly one stable matching. The proof of the claim is by contradiction. Assume that $\nu \neq \mu$ is a stable matching. Let s be the minimal index such that $\mu(w_s) \neq \nu(w_s)$. Since ν is individually rational, then $\nu(w_s) \subseteq A_s(P)$. Let us assume that $\mu(w_s) P_{w_s} \nu(w_s)$. Thus, there exists either $f \in \mu(w_s) \setminus \nu(w_s)$ or $\mu(w_s) \subset \nu(w_s)$. In the first case, the minimality of s implies that either $|\nu(f)| < q_f$ or that there exists $t > s$ with $w_t \in x(f)$. Then (f, w_s) blocks x yielding a contradiction.
In the case that $\mu(w_s) \subset \nu(w_s)$, by assumption $\mu(w_s) P_{w_s} \nu(w_s)$. Additionally $|\mu(w_s)| < q_{w_s}$, since ν is individually rational. Let $f \in \nu(w_s) \setminus \mu(w_s)$. The minimality of s implies $f \in A_s(P)$; thus, (f, w_s) blocks μ yielding a contradiction. □

Proof of Theorem 1. We prove the claim by contradiction. Let us assume that there exists a nonempty set of agents $V' \subset V$, P and $P'_{V'} = (P'_v)_{v \in V'}$ such that $\mu^W(P'_{V'}, P_{V \setminus V'})(v) R_v \mu^W(P)(v)$ for every $v \in V'$ and $\mu^W(P'_{V'}, P_{V \setminus V'})(v') P_{v'} \mu^W(P)(v')$ for some $v' \in V'$.

The strategy for the proof is as follows. First, we show that there exists a profile of acyclical preferences that guarantees the same outcome as $P'_{V'}$ to all members of the coalition. Then, we use this result, to show that if the preferences of the firms are acyclical, no group of workers can profitably benefit from a deviation from truth-telling. Then, we observe that there is no loss of generality, assuming that only a coalition of firms is deviating. Finally, we prove that in this case, the preferences of the firms have a cycle, which leads to a contradiction.

Let $P''_{V'} = (P''_v)_{v\in V'}$ be such that for every $v \in V'$, P''_v is responsive and coincides with P_v on the subsets of $\mu^W(P'_{V'}, P_{V\setminus V'})(v)$ and ranks all subsets containing agents in $V \setminus \mu^W(P'_{V'}, P_{V\setminus V'})(v)$ as unacceptable. Formally, for every $f \in F \cap V'$, for all $W', W'' \subseteq \mu^W(P'_{V'}, P_{V\setminus V'})(f)$, let $W'P''_f W''$ if and only if $W'P_f W''$; if $W' \cap W \setminus \mu^W(P'_{V'}, P_{V\setminus V'})(f) \neq \emptyset$, let $\emptyset P''_f W'$. For every $w \in W \cap V'$, for all $F', F'' \subseteq \mu^W(P'_{V'}, P_{V\setminus V'})(w)$, let $F'P''_w F''$ if and only if $F'P_w F''$; if $F' \cap W \setminus \mu^W(P'_{V'}, P_{V\setminus V'})(w) \neq \emptyset$, let $\emptyset P''_w F'$. The profiles $P''_F = \left(P''_{V'\cap F}, P_{F\setminus V'\cap F}\right)$ and $P''_W = \left(P''_{V'\cap W}, P_{W\setminus V'\cap W}\right)$ are responsive. We have $\mu^W(P''_{V'}, P_{V\setminus V'})(v) = \mu^W(P'_{V'}, P_{V\setminus V'})(v)$ for all $v \in V'$. Let $w_1, w_2, ..., w_{|W|}$ be an order used to generate $\mu(P) = \mu^W(P)$ as an adjusted serial dictatorship. For every $f \in F$, preferences P'_f and P''_f coincide on the set of mutually acceptable workers and $A\left(f, P''_f\right) \subseteq A(f, P_f)$. It follows that P''_F is acyclical and $w_1, w_2, ..., w_{|W|}$ can be used to generate an adjusted serial dictatorship leading to $\mu\left(P''_{V'}, P_{V\setminus V'}\right)$.

From now on, let $\mu = \mu(P)$ and let $\nu = \mu^W(P''_{V'}, P_{V\setminus V'})$.

First of all, we prove $V' \cap F \neq \emptyset$. The proof of this claim is by contradiction. Assume that $V' \subseteq W$, and let s be the minimum integer such that $\nu(w_s) \neq \mu(w_s)$. Since all workers with an index lower than s are matched to the same firms under μ and under ν and $V' \subseteq F$, we have $A_s\left(P''_{V'}, P_{V\setminus V'}\right) = A_s(P)$. If $w_s \in V'$ $\mu(w_s) P_{w_s} \nu(w_s)$, which yields a contradiction. Otherwise, $\mu(w_s) = \nu(w_s)$, which also yields a contradiction.

Next, we prove that there is no loss of generality in assuming that $V' \subseteq F$. More precisely, we prove that $\mu^W\left(P''_{V'\cap F}, P_{V\setminus V'\cap F}\right)(v) R_v \mu^W(P)(v)$, for each $v \in V'$. Thus if coalition V' can manipulate μ^W, coalition $V' \cap F$ can also manipulate μ^W. Notice that the claim is true for all $v' \in F$, since $\mu^W\left(P''_{V'\cap F}, P_{V\setminus V'\cap F}\right)(f) = \nu(f)$ for each $f \in V' \cap W$. We complete the proof of the claim by contradiction. Notice that if $\mu^W(P) P_w \mu^W\left(P''_{V'\cap F}, P_{V\setminus V'\cap F}\right)(w)$ for some $w \in V' \cap W$, then $\mu^W\left(P''_{V'}, P_{V\setminus V'}\right) P_w \mu^W\left(P''_{V'\cap F}, P_{V\setminus V'\cap F}\right)(w)$ as well. Let s be the minimum integer such that $\mu^W\left(P''_{V'}, P_{V\setminus V'}\right)(w_s) P_{w_s} \mu^W\left(P''_{V'\cap F}, P_{V\setminus V'\cap F}\right)(w_s)$. Since $\mu^W\left(P''_{V'\cap F}, P_{V\setminus V'\cap F}\right)(f) = \nu(f)$ for each $f \in V' \cap W$, $w_s \in W \cap V'$. Since all workers with an index lower than s are matched to the same firms under $\mu^W\left(P''_{V'\cap F}, P_{V\setminus V'\cap F}\right)(w_s)$ and under $\mu^W\left(P''_{V'}, P_{V\setminus V'}\right)$, $A_s\left(P''_{V'\cap F}, P_{V\setminus V'\cap F}\right) = A_s\left(P''_{V'}, P_{V\setminus V'}\right)$; thus, $\mu^W\left(P''_{V'\cap F}, P_{V\setminus V'\cap F}\right)(w_s) R_{w_s} \mu^W\left(P''_{V'}, P_{V\setminus V'}\right)$, which yields a contradiction.

Finally, assume $V' \subseteq F$. Since P''_F and P_F are acyclical, the matchings μ and ν coincide with the firm optimal stable matchings in markets (F, W, P) and $\left(F, W, P''_{V'\cap F}, P_{V\setminus V'\cap F}\right)$, respectively. Consider the deferred acceptance algorithm where firms propose to workers in market (F, W, P) (see Alkan, 1999). Since the stable set is a singleton, the algorithm yields $\mu^W(P)$ as an outcome. Notice that for all $f \in F$, $\mu^W\left(P''_{V'}, P_{V\setminus V'}\right)(f) R_f \mu^W(P)(f)$. There exists a firm $f \in F$ such that $\nu(f) P_f \mu(f)$ and $w \in W$ such that $f \in \nu(w) \setminus \mu(w)$,

wP_fw' for some $w' \in \mu(f) \setminus \nu(f)$.[8] Let f be the first of such firms to be definitively matched to $\mu(f)$ in the deferred acceptance algorithm. In a previous step of the algorithm leading to μ, f has been rejected by some $w_0' \in \nu(f) \setminus \mu(f)$. Worker w_0 rejects f in favor of $f_1' \in F$. Notice $f_1' \neq f$. Since in the deferred acceptance algorithm leading to $\mu^W\left(P''_{V'}, P_{V\setminus V'}\right)$, f_1' had not applied to w_0, who is acceptable to it, it must be the case that in a previous stage, f_1' had been rejected by w_1 such that $w_1'P_{f_1'}w_0'$ in favor of some $f_2' \in F$ that did not apply to w_1' along the deferred acceptance algorithm yielding ν. Let $k \geq 2$, using the same argument, it can be shown that there exist $f_k' \notin \nu(w_k') \setminus (\mu(w_k'))$, $f_k' \neq f_{k-1}'$ and w_k' such that w_k' rejects f_k' in favor of f_{k+1}', where $f_{k+1}' \neq f_k'$ in the deferred acceptance algorithm, leading to μ and $w_k'P_{f_k'}w_{k-1}'$. Let r be the minimum integer such that $w_r' = w_{r-j}'$ for some $j \leq r$. Such an integer exists because the set of agents is finite. Without loss of generality assume $j = r$. Notice that, by construction, $w_k' \neq w_{k+1}'$, for all $k = 1, 2, ..., r-1$. Let $F' = \{f \in F : \exists i, k, 0 \leq i \leq r, , 0 \leq k \leq r, i \neq k, f = f_i' = f_k'\}$. For all $f \in F'$ let $i_1(f) = \min\{i : w = f_i'\}$ and let $i_2(f) = \max\{i : f = f_i'\}$. Let $I = \{i : \exists f \in F', i_1(f) \leq i < i_2(f)\}$. Notice that $|I| \geq 2$. It follows that $\{f_i'\}_{i \notin I}$ and $\{w_i'\}_{i \notin I}$ form a cycle, which yields a contradiction. □

Proof of Lemma 1. Let $f_0, f_1, ..., f_T$, $w_0, w_1, ..., w_T$ such that $w_iP_{f_i}w_{i-1}$ for $i = 0, ..., T$, where $f_{-1} = f_T$. Set $P_{w_0} : \{f_1\}, \{f_0\}$, $P_{w_1} : \{f_2\}, \{f_1\}$, and set $P_{w_i} : \{f_{i+1}\}, \{f_i\}$ for $i = 1, 2, ..., T-1$. For all $w \notin \{w_0, w_1, ..., w_T\}$, let P_w such that $A(w) = \emptyset$. Let μ^W be the worker-optimal stable mechanism. We have $\mu^W(P)(w_i) = \{f_{i+1}\}$ for $i = 0, 1, ..., T$. Then, $P_{f_1}' = \{w_1\}$ is a profitable deviation from the truth-telling strategy for f_1. □

Proof of Proposition 2. Let P_W in the proof of Proposition 1. By contradiction, assume that there exists a stable and group strategy-proof mechanism, φ. There are exactly two stable matchings, $\mu^W(P)$ and $\mu^F(P)$, where$\mu^W(P)(w_i) = \{f_{i+1}\}$ for $i = 0, 1, ..., T$ and $\mu^F(P)(w_i) = \{f_i\}$ for $i = 0, 1, ...,$T. From the proof of Proposition 1, it follows that $\varphi(P) = \mu^F(P)$. In this case, $P_{w_0}' = \{f_1\}$ is a profitable deviation from the truth-telling strategy for w_0. □

Proof of Lemma 2. Let $f_0, f_1, ..., f_T$, $w_0, w_1, ..., w_T$ such that $w_i \succ_{f_i} w_{i-1}$ for $i = 0, ..., T$, where $f_{-1} = f_T$. Set $q_f = 1$ for all $f \in F$. Set $P_{w_0} : \{f_1, f_0\}, \{f_1\}, \{f_0\}$, $P_{w_1} : \{f_2\}, \{f_1\}$, and set $P_{w_i} : \{f_{i+1}\}, \{f_i\}$ for $i = 1, 2, ..., T-1$. For all $w \notin \{w_0, w_1, ..., w_T\}$, let P_w such that $A(w) \subseteq F \setminus \{f_0, f_1, ..., f_T\}$. Let φ be a stable mechanism. We have $\varphi(P, q)(f_i) = \{w_i\}$ for $i = 0, 1, ..., T$. Let $P_{w_0}' = \{f_1\}$. Then, $\{f_1\} = \varphi\left(P_{w_0}', P_{-w_0}, q\right)(w_0)\, P_{w_0}\, \{f_0\} = \varphi(P)(w_0)$,which implies the claim. □

Aknowledgements

Both authors acknowledge financial support from Ministerio Economía y Competitividad (Spain) under project ECO2014_57442_P and from Fondecyt under

[8]Since μ is stable, for all w such that $f \in \nu(w) \setminus \mu(w)$, $|\mu(w)| = q_w$.

project No. 1151230. Romero-Medina acknowledges the financial support from Ministerio Economía y Competitividad (Spain) MDM 2014-0431, and Comunidad de Madrid, MadEco-CM (S2015/HUM-3444) and Triossi acknowledges the financial support from the Institute for Research in Market Imperfections and Public Policy, ICM IS130002, Ministerio de Economía, Fomento y Turismo. Triossi acknowledges the hospitality of Collegio Carlo Alberto while writing this paper.

References

Abdulkadiroğlu, A., Sönmez, T., 2003. School Choice: A Mechanism Design Approach. Amer. Econ. Rev. 93. 729–747.

Abdulkadiroğlu, A., Sönmez, T., 2013. Matching Markets: Theory and Practice. Advances in Economics and Econometrics Theory and Applications, Tenth World Congress, Volume II.

Alkan, A., 1999. On the properties of stable many-to-many matchings under responsive preferences. In: Alkan, A., Aliprantis, C.D., Yannelis, N.C. (Eds.), Current Trends in Economics: Theory and Applications. In: Studies in Economic Theory, vol. 8. Springer, Berlin, Heidelberg, pp. 29–39.

Balinski, M., Sönmez, T., 1999. A Tale of Two Mechanisms: Student Placement. J. Econ. Theory. 84, 73–94.

Barberá,S., Berga, D., Moreno, B., 2016. Group Strategy-proofness in Private good Economies. Amer. Econ. Rev. 106(4), 1073–1099.

Budish, E., 2011. The Combinatorial Assignment Problem: Approximate Competitive Equilibrium from Equal Incomes. J. Polit. Economy. 119, 1061–1103.

Ehlers, L., Klaus, B., 2003. Coalitional strategy-proof and resource-monotonic solutions for multiple assignment problems. Soc. Choice Welfare 21, 265–280.

Ergin, H. I., 2002. Efficient Resource Allocation on the Basis of Priorities. Econometrica. 70, 2489–2497.

Figueroa, N., Lafortune, J., Saenz, A., 2017. Do you like me enough? The impact of restricting preferences ranking in a university matching process. Mimeo.

Hatfield, J.W., Kojima, F., 2009. Group incentive compatibility for matching with contracts. Games Econ. Behav. 67, 745–749.

Jiao, Z., Tian, G., 2017. The Blocking Lemma and strategy-proofness in many-to-many matchings. Games Econ. Behav. 102, 44–55.

Kesten, O., 2012. On Two Kinds of Manipulation for School Choice Problems. Econ. Theory. 51, 677–693.

Kojima, F., 2013. Efficient Resource Allocation under Multi-unit Demand. Games Econ. Behav. 82, 1–14.

Roth, A. E., 1991. A natural experiment in the organization of entry level labor markets: regional markets for new physicians and surgeons in the UK. Amer. Econ. Rev. 81 (3), 415–440.

Roth, A.E., Sotomayor, M., 1990. Two-Sided Matching: A Study in Game-Theoretic Modeling and Analysis, Cambridge, Cambridge University Press.

Romero-Medina, A., Triossi, M., 2013a. Games with Capacity manipulation, Incentives and Nash Equilibria. Soc. Choice Welfare. 41, 701–720.

Romero-Medina, A., Triossi, M., 2013b. Acyclicity and Singleton Cores in Matching Markets. Econ. Letters. 118, 237–239.

Romero-Medina, A., Triossi, M., 2017. Group Strategy-Proof Stable Mechanisms in Priority-Based Resource Allocation Under Multi-Unit Demand. Available at SSRN: https://ssrn.com/abstract=3030175.

Schummer, J., Abizada, A., 2017. Incentives in landing slot problems. J. Econ. Theory. 170, 29–55.

Schummer, J., Vohra, R.V., 2013. Assignment of Arrival Slots. Amer. Econ. J.: Microeconomics. 5, 164–185.

Sönmez, T., 1997. Manipulation via capacities in two-sided matching markets. J. Econ. Theory 77, 197–204.

Sönmez, T., Ünver, U., 2010. Course Bidding at Business Schools. Int. Econ. Rev. 51(1). 99–123.